E2.5 Signals and Linear Systems.

Tutorial Sheet 9 - Solutions

1) (a) $f[k] = k^2 \gamma^k u[k]$

Given $\gamma^k u[k] \Leftrightarrow \frac{z}{z-\gamma}$

Use the multiplication by k property; which states

$$k\, x[k]\, u[k] \Leftrightarrow -z \frac{d}{dz} X[z]$$

Let $x[k] = \gamma^k u[k]$,

$$\frac{d}{dz} X[z] = \frac{d}{dz}\left(\frac{z}{z-\gamma}\right) = \frac{(z-\gamma) - z}{(z-\gamma)^2}$$

$$= -\frac{\gamma}{(z-\gamma)^2}$$

$$\therefore\ k \gamma^k u[k] \Leftrightarrow \frac{z\gamma}{(z-\gamma)^2}$$

Apply the above again

$$k^2 \gamma^k u[k] \Leftrightarrow \frac{\gamma z (z+\gamma)}{(z-\gamma)^3}$$

(b) $f[n] = n^3 u[n]$

We use results from (a), apply the "multiply by n" property one more time

$$n^3 \gamma^n u[n] \Leftrightarrow -z \frac{d}{dz}\left[\frac{\gamma z(z+\gamma)}{(z-\gamma)^3}\right]$$

$$\Leftrightarrow \frac{\gamma z (z^2 + 4\gamma z + \gamma^2)}{(z-\gamma)^4}$$

Let $\gamma = 1$,

$$n^3 u[n] \Leftrightarrow \frac{z(z^2 + 4z + 1)}{(z-1)^4}$$

1) (c) $f[k] = a^k \{u[k] - u[k-m]\}$

$$= a^k u[k] - a^m a^{(k-m)} u[k-m]$$

$$\therefore F[z] = \frac{z}{z-a} - a^m \frac{z}{z-a} z^{-m}$$

$$= \frac{z}{z-a}\left[1 - \left(\frac{a}{z}\right)^m\right] //$$

2) Solution is found in Lathi's book, p. 510.

$x[n] = n\{u[n] - u[n-6]\}$

$= n u[n] \; - n\, u[n-6] \leftarrow$ Rearrange this!

$= n\, u[n] - (n-6+6)\, u[n-6]$

$= n\, u[n] - (n-6) u[n-6] - 6 u[n-6].$

Use right-shift property

$$u[n-6] \Longleftrightarrow \frac{1}{z^6}\frac{z}{(z-1)} = \frac{1}{z^5(z-1)}$$

$$(n-6)\, u[n-6] \Longleftrightarrow \frac{1}{z^6}\frac{z}{(z-1)^2} = \frac{1}{z^5(z-1)^2}.$$

$$\therefore X[z] = \frac{z}{(z-1)^2} - \frac{1}{z^5(z-1)^2} - \frac{6}{z^5(z-1)}$$

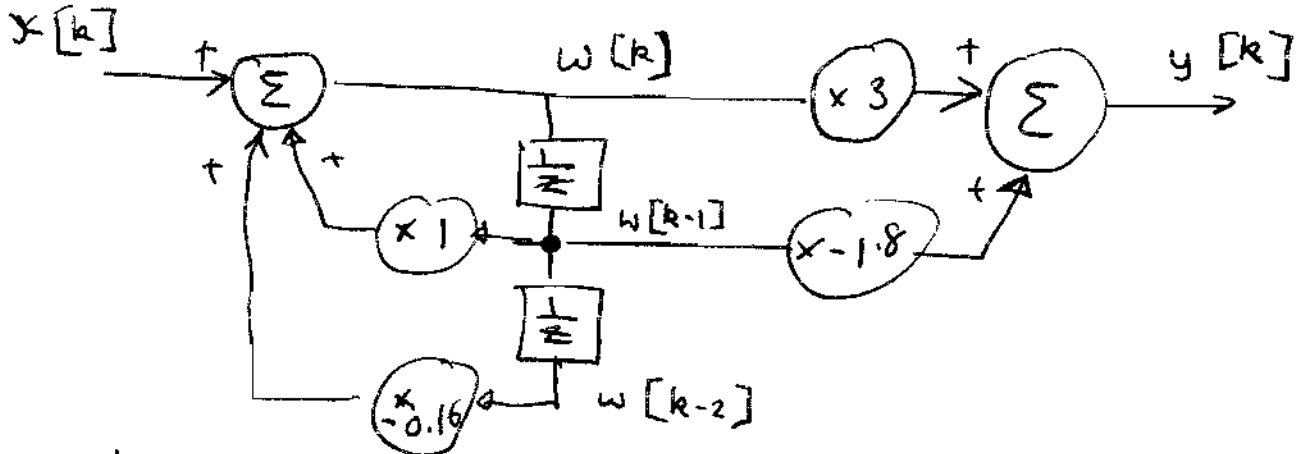
$$= \frac{z^6 - 6z + 5}{z^5(z-1)^2} //$$

3) (a)

$$H(z) = \frac{z(3z - 1.8)}{z^2 - z + 0.16}$$

$$= \frac{3z^2 - 1.8z}{z^2 - z + 0.16}$$

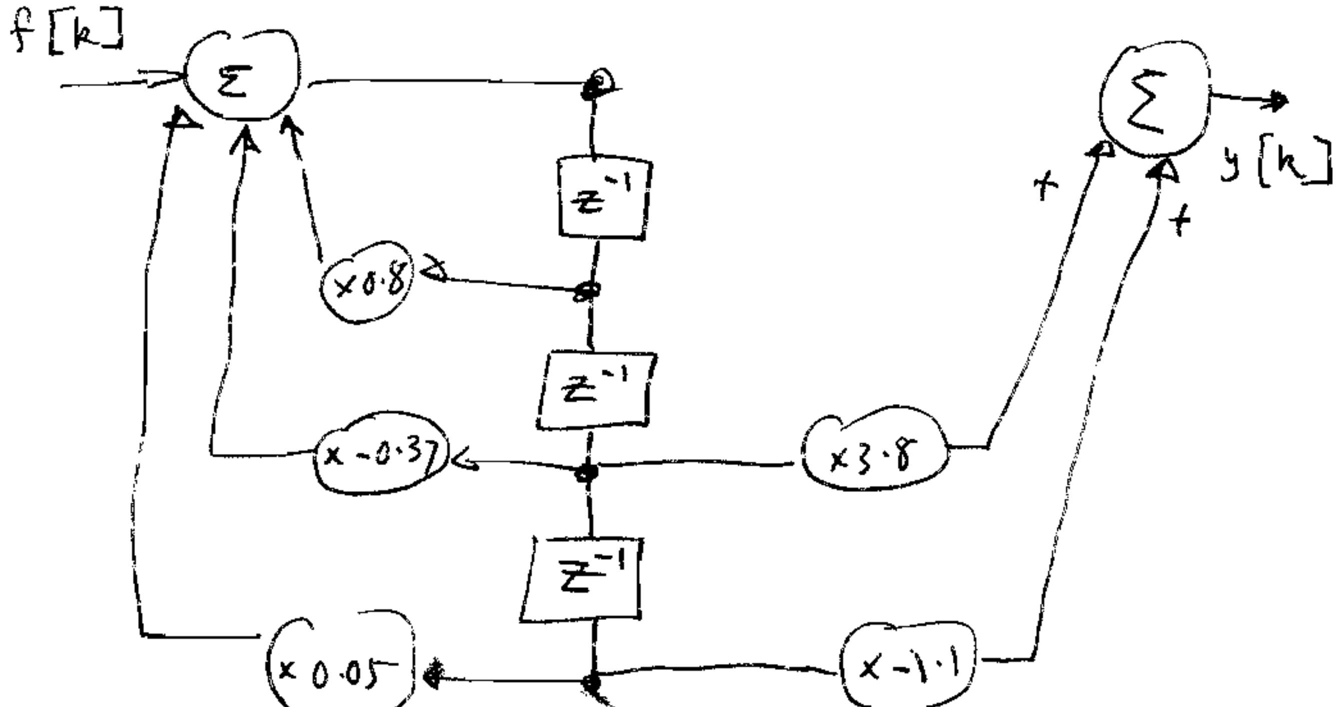
$$= \frac{3 - 1.8z^{-1}}{1 - z^{-1} + 0.16z^{-2}}.$$



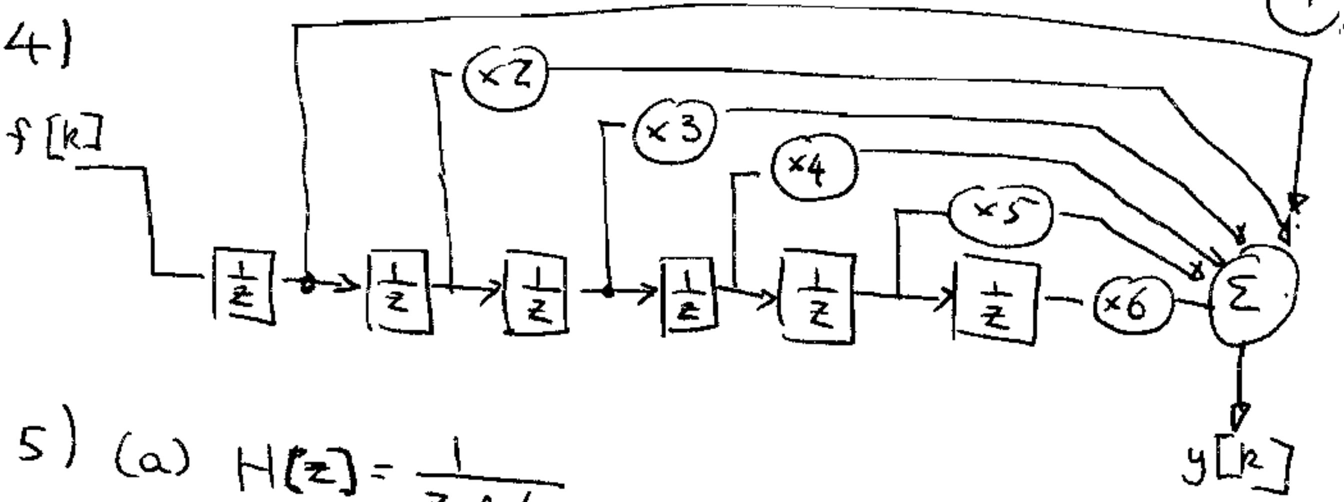
(b)

$$H(z) = \frac{3.8z - 1.1}{(z-0.2)(z^2 - 0.6z + 0.25)}$$

$$= \frac{3.8z^{-2} - 1.1z^{-3}}{1 - 0.8z^{-1} + 0.37z^{-2} - 0.05z^{-3}}$$



4)



5) (a) $H[z] = \frac{1}{z-0.4}$

$$\therefore H[e^{j\Omega}] = \frac{1}{e^{j\Omega}-0.4} = \frac{1}{\cos\Omega - 0.4 + j\sin\Omega}.$$

$$|H[e^{j\Omega}]|^2 = H H^* = \frac{1}{(e^{j\Omega}-0.4)(e^{-j\Omega}-0.4)} = \frac{1}{1.16 - 0.8\cos\Omega}$$

$$|H[e^{j\Omega}]| = \frac{1}{\sqrt{1.16 - 0.8\cos\Omega}} \quad //$$

and $\angle H[e^{j\Omega}] = -\tan^{-1}\frac{\sin\Omega}{(\cos\Omega - 0.4)}$ //

5) (b) $H(z) = \frac{3z^2 - 1.8z}{z^2 - z + 0.16}$

$$\therefore H[e^{j\Omega}] = \frac{3e^{2j\Omega} - 1.8e^{j\Omega}}{e^{2j\Omega} - e^{j\Omega} + 0.16}$$

$$= \frac{(3\cos 2\Omega - 1.8\cos\Omega) + j(3\sin 2\Omega - 1.8\sin\Omega)}{(\cos 2\Omega - \cos\Omega + 0.16) + j(\sin 2\Omega - \sin\Omega)}.$$

5) (b) continue .....

$$\left|H[e^{j\Omega}]\right|^2 = \left[\frac{3e^{2j\Omega} - 1.8e^{j\Omega}}{e^{2j\Omega} - e^{j\Omega} + 0.16}\right]\left[\frac{3e^{-2j\Omega} - 1.8e^{-j\Omega}}{e^{-2j\Omega} - e^{-j\Omega} + 0.16}\right]$$

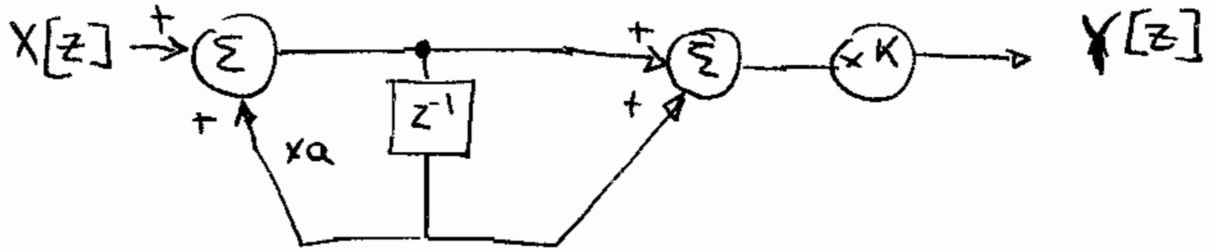
$$= \frac{12.24 - 10.8\cos\Omega}{2.0256 - 2.32\cos\Omega + 0.32\cos 2\Omega}$$

$$\therefore \left|H[e^{j\Omega}]\right| = \left[\frac{12.24 - 10.8\cos\Omega}{2.0256 - 2.32\cos\Omega + 0.32\cos 2\Omega}\right]^{\frac{1}{2}}$$ //

$$\angle H[e^{j\Omega}] = \tan^{-1}\left(\frac{3\sin 2\Omega - 1.8\sin\Omega}{3\cos 2\Omega - 1.8\cos\Omega}\right)$$

$$- \tan^{-1}\left(\frac{\sin 2\Omega - \sin\Omega}{\cos 2\Omega - \cos\Omega + 0.16}\right)$$ //

6) (a) $H(z) = K \frac{z+1}{z-a}$.



(b) $H[1] = \frac{2K}{1-a} = 1 \Rightarrow K = \frac{1-a}{2}$

Hence $H(z) = \left(\frac{1-a}{2}\right)\left(\frac{z+1}{z-a}\right)$

$$H[e^{j\Omega}] = \frac{1-a}{2}\left(\frac{\cos\Omega + 1 + j\sin\Omega}{\cos\Omega - a + j\sin\Omega}\right).$$

$$\therefore |H[e^{j\Omega}]| = \left(\frac{1-a}{2}\right)\sqrt{\frac{2(1+\cos\Omega)}{1+a^2-2a\cos\Omega}}$$

$$\angle H[e^{j\Omega}] = \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega+1}\right) - \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega - a}\right)$$

For $a = 0.2$,

$$|H[e^{j\Omega}]| = 0.4\sqrt{\frac{2(1+\cos\Omega)}{1.04 - 0.4\cos\Omega}}$$

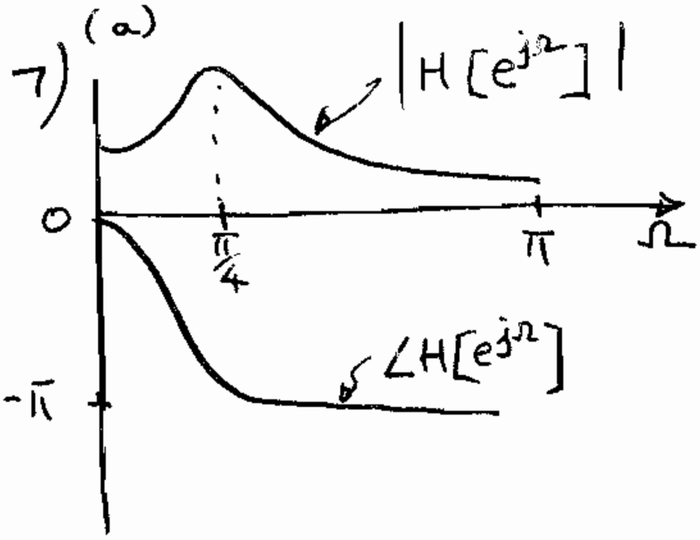
For 3 dB bandwidth, $|H|^2 = 1/2$.

$$\therefore \frac{1}{2} = 0.4^2\left[\frac{2(1+\cos\Omega)}{1.04-0.4\cos\Omega}\right]$$

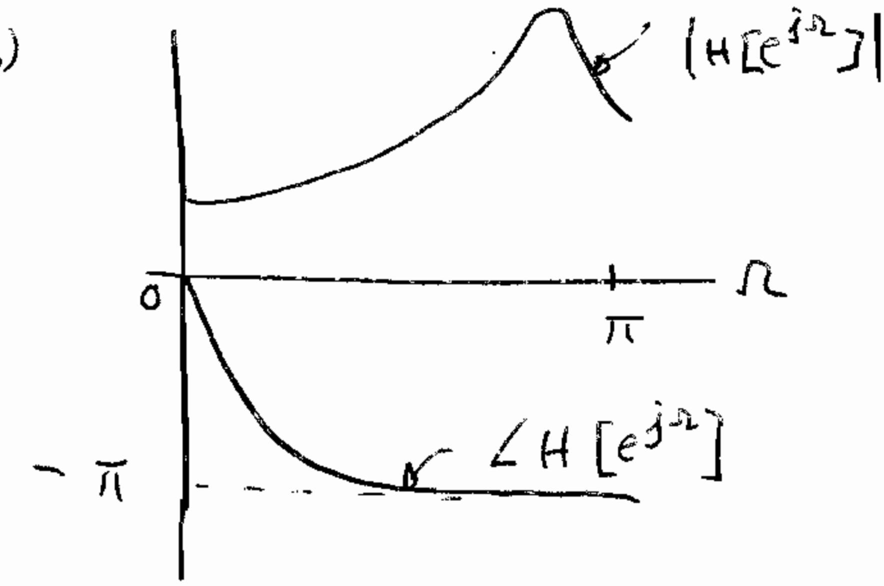
$$\Rightarrow \Omega = 1.176 \,//$$

Hence $B = \frac{\omega}{2\pi} = \frac{\Omega}{2\pi T} = \frac{1.176}{2\pi T} = \frac{0.187}{T}$ Hz //

7) (a)



(b)



8) Sampling Period $= \frac{1}{40{,}000} = 25\mu s$, $f_s = 40kHz$.

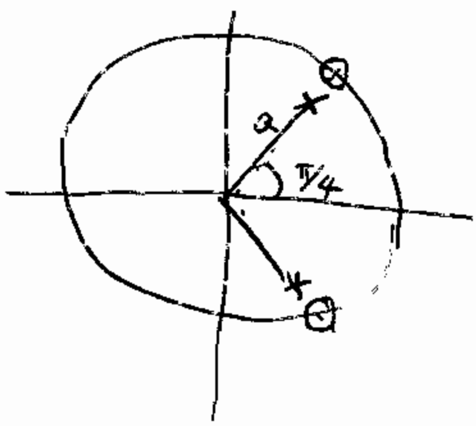
$$\Omega = \omega T = 2\pi \times 5000 \times 25 \times 10^{-6}$$
$$= \pi/4.$$

$\therefore$ Zeros on unity circle at $\pm \pi/4$

Poles at $a e^{\pm j\pi/4}$

$$H[z] = K \frac{(z - e^{j\pi/4})(z - e^{-j\pi/4})}{(z - a e^{j\pi/4})(z - a e^{-j\pi/4})}$$

$$= K \frac{(z^2 - \sqrt{2} z + 1)}{z^2 - \sqrt{2} a z + a^2}$$



when $\omega = 0$, $H[1] = 1$

$\therefore K = 1.707(1 + a^2 - \sqrt{2} a)$

9) (a) $$H[z] = \frac{z - \frac{1}{r}}{z - r} \qquad r \leq 1 .$$

$$\left|H[e^{j\Omega}]\right|^2 = \frac{\left(e^{j\Omega} - \frac{1}{r}\right)\left(e^{-j\Omega} - \frac{1}{r}\right)}{\left(e^{j\Omega} - r\right)\left(e^{-j\Omega} - r\right)}$$

$$= \frac{1 + \frac{1}{r^2} - \frac{2}{r}\cos\Omega}{1 + r^2 - 2r\cos\Omega} = \frac{1}{r^2}$$

$\therefore \left|H[e^{j\Omega}]\right| = \frac{1}{r}$ which is constant with Ω.

(b) $$H[z] = \frac{\left(z - \frac{1}{r}e^{j\theta}\right)\left(z - \frac{1}{r}e^{-j\theta}\right)}{\left(z - re^{j\theta}\right)\left(z - re^{-j\theta}\right)} \qquad r \leq 1$$

$$= \frac{z^2 + \frac{1}{r^2} - z\frac{1}{r}\left(e^{-j\theta} + e^{j\theta}\right)}{z^2 + r^2 - z\,r\left(e^{-j\theta} + e^{j\theta}\right)}$$

$$= \frac{z^2 + \frac{1}{r^2} - \frac{2}{r}\cos\theta\ z}{z^2 + r^2 - 2r\cos\theta\ z}$$

$$\left|H[e^{j\Omega}]\right|^2 = \frac{\left(e^{2j\Omega} + \frac{1}{r^2} - \frac{2}{r}\cos\theta\ e^{j\Omega}\right)\left(e^{-2j\Omega} + \frac{1}{r^2} - \frac{2}{r}\cos\theta\ e^{-j\Omega}\right)}{\left(e^{2j\Omega} + r^2 - 2r\cos\theta\ e^{j\Omega}\right)\left(e^{-2j\Omega} + r^2 - 2r\cos\theta\ e^{-j\Omega}\right)}$$

$$= \frac{1}{r^4}$$

$$\therefore \left|H[e^{j\Omega}]\right| = \frac{1}{r^2} \quad //$$